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INTEGRATION OF THE SOLUTION OF THE LING HEAT PROBLEM USING FINITE FUNCTIONS

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It is proposed that a step or piecewise linear approximation of the intensity of the heat flux be used for calculating convolution integrals in the solution of the Ling heat problem.

Introduction. In 1965, Professor Ling and his assistant Mow from Rensselaer Polytechnic Institute (USA) obtained a solution of a plane quasistationary problem of heat conduction for a half-space that is heated, on a free surface, by a fast-moving linear distributed heat flux [1]. In connection with successful further application of this solution to modeling thermal regimes of various tribotechnical processes (determination of the coefficients of the distribution of heat between the elements of friction pairs [2–5], grinding [6, 7], and heat generation during the sliding of a wheel over a rail [8–10], etc.) the above problem is known in scientific and technical literature as the "Ling problem" [11]. In spite of a more than 30-year-long history the Ling problem still attracts attention, first, due to an increasingly greater number of new fields of its application, second, due to the simplicity of the representation of the solution, and, third, due to the necessity of investigating the solution (integration) for different heat-flux densities. This is precisely the reason for the appearance of the given work.

We also note that the Ling problem is a particular (for large values of the Péclet number) case of the well-known Jaeger problem [12].

Formulation and Solution of the Ling Heat Problem. Let us consider the plane boundary-value problem of quasistationary heat conduction

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{V}{k}\frac{\partial T}{\partial x}, \quad |x| < \infty, \quad y > 0, \tag{1}$$

$$K \frac{\partial T}{\partial y}\bigg|_{y=0} = \begin{cases} -q(x), & 0 \le x \le 2a, \\ 0, & -\infty < x < 0 \cup 2a < x < \infty, \end{cases} \tag{2}$$

$$T \to 0 \quad \text{for} \quad \sqrt{x^2 + y^2} \to \infty. \tag{3}$$

For heat problems of friction we assume that the intensity of the frictional heat flux q is equal to the specific power of the friction forces [11]

$$q(x) = fVp(x). \tag{4}$$

We denote

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$$\xi = \frac{x}{2a}, \quad \eta = \frac{y}{d}, \quad p^* = \frac{p}{p_0}, \quad d = \sqrt{\frac{2ak}{V}}, \quad \mathrm{Pe} = \frac{Va}{2k}. \tag{5}$$

With account for equality (4) and notation (5) Eq. (1) and boundary conditions (2)–(3) will take the form

$$\frac{1}{4\mathrm{Pe}}\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} = \frac{\partial T}{\partial \xi}, \quad |\xi| < \infty, \quad \eta > 0; \tag{6}$$

$$\left.\frac{\partial T}{\partial \eta}\right|_{\eta=0} = \begin{cases} -\Lambda p^*(\xi), & 0 \le \xi \le 1, \\ 0, & -\infty < \xi < 0 \cup 1 < \xi < \infty; \end{cases} \tag{7}$$

$$T \to 0 \quad \text{for} \quad \sqrt{\xi^2 + \eta^2} \to \infty, \tag{8}$$

where

$$\Lambda = \frac{fVp_0 d}{K}. \tag{9}$$

Let us consider the case of a fast-moving heat source where Pe > 5 [1]. The gradient of the heat flux in the direction of motion can be disregarded, and the heat-conduction equation (6) will take the form

$$\frac{\partial^2 T}{\partial \eta^2} = \frac{\partial T}{\partial \xi}, \quad |\xi| < \infty, \quad \eta > 0. \tag{10}$$

Equation (10) and boundary conditions (7)–(8) are a mathematical representation of the Ling heat problem.

The solution of the boundary-value problem of quasistationary heat conduction (10) and (7)–(8) is obtained using the integral Fourier transformation [1]

$$T(\xi, \eta) = \frac{\Lambda}{\sqrt{\pi}} \int_0^b G(\xi - \tau, \eta)\, p^*(\tau)\, d\tau, \quad |\xi| < \infty, \quad 0 \le \eta < \infty, \tag{11}$$

$$G(\xi, \eta) = \frac{1}{\sqrt{\xi}} \exp\left(-\frac{\eta^2}{4\xi}\right), \quad b = \begin{cases} 0, & -\infty < \xi < 0, \\ \xi, & 0 \le \xi \le 1, \\ 1, & 1 < \xi < \infty. \end{cases} \tag{12}$$

To calculate the integral in the right-hand side of relation (11), it is proposed in [13] that the function $p^*(\tau)$ be expanded in Fourier sine- or cosine-series with subsequent accurate determination of the coefficients of the expansion and summation. However the expressions for calculating the coefficients of the Fourier series are so complicated, and, what is more, are represented in complex form, that we could not find a single work where the given procedure was used. Most frequently the authors restricted themselves to numerical integration of (11) for specific distributions of the contact pressure $p^*(\tau)$.

We propose a procedure for integrating the solution (11) in the case of an arbitrary smooth function $p^*(\tau)$ using the properties of finite step and piecewise-linear functions [14].

Step Approximation. We introduce on the segment [0, b] the uniform grid

$$0=\tau_0<\tau_1<\ldots<\tau_{n-1}<\tau_n=b\,,\quad \tau_i=i\delta\tau\,,\quad \delta\tau=b/n\,,\quad i=0,1,2,\ldots,n\,. \tag{13}$$

Let

$$p^*(\tau)\cong\tilde{p}^*(\tau)=\sum_{i=1}^{n}p_i^*\varphi_i(\tau)\,, \tag{14}$$

where

$$p_i^*=p^*\left(\frac{\tau_{i-1}+\tau_i}{2}\right);\quad \varphi_i(\tau)=\begin{cases}1\,, & \tau\in[\tau_{i-1},\tau_i]\,,\\ 0\,, & \tau\notin[\tau_{i-1},\tau_i]\,.\end{cases} \tag{15}$$

The uniform error of the approximation (14) and (15) is $O(\delta\tau)$ [14].

Having substituted the function $\tilde{p}^*(\tau)$ of (14) into the solution (11) under the integration sign, we obtain

$$T(\xi,\eta)=\frac{\Lambda}{\sqrt{\pi}}\sum_{i=1}^{n}p_i^*\,\tilde{G}_i^{(0)}(\xi,\eta)\,H(\xi)\,,\quad -\infty<\xi<\infty\,,\quad 0\le\eta<\infty\,, \tag{16}$$

where

$$\tilde{G}_i^{(0)}(\xi,\eta)=\int_{\tau_{i-1}}^{\tau_i}G(\xi-\tau,\eta)\,d\tau\,,\quad \xi\ge\tau_i\,,\quad i=1,2,\ldots,n\,. \tag{17}$$

With account for Green's functions $G(\xi,\eta)$ of (12), upon integration by parts the functions $\tilde{G}_i^{(0)}(\xi,\eta)$ of (17) will take the form

$$\tilde{G}_i^{(0)}(\xi,\eta)=-2\sqrt{\xi-\tau_i}\,\exp\left[-\frac{\eta^2}{4(\xi-\tau_i)}\right]+2\sqrt{\xi-\tau_{i-1}}\,\exp\left[-\frac{\eta^2}{4(\xi-\tau_{i-1})}\right]-$$

$$-\eta\sqrt{\pi}\left[\operatorname{erf}\left(\frac{\eta}{2\sqrt{\xi-\tau_i}}\right)-\operatorname{erf}\left(\frac{\eta}{2\sqrt{\xi-\tau_{i-1}}}\right)\right],\quad \xi\ge\tau_i\,,\quad i=1,2,\ldots,n\,. \tag{18}$$

For $\eta=0$, from relation (18) we have

$$\tilde{G}_i^{(0)}(\xi,0)=2\left(\sqrt{\xi-\tau_{i-1}}-\sqrt{\xi-\tau_i}\right),\quad \xi\ge\tau_i\,,\quad i=1,2,\ldots,n\,. \tag{19}$$

For $\eta=0$ and with account for the functions $\tilde{G}_i^{(0)}(\xi,0)$ of (19) the solution (16) yields the distribution of the temperature of the half-space surface $T(\xi)\equiv T(\xi,0)$

$$T(\xi)=\frac{\Lambda}{\sqrt{\pi}}\sum_{i=1}^{n}p_i^*\,\tilde{G}_i^{(0)}(\xi,0)\,H(\xi)\,,\quad -\infty<\xi<\infty\,, \tag{20}$$

which was obtained earlier [8].

Piecewise-Linear Approximation. Let each node τ_i, i = 0, 1, 2, ..., n of the grid (13) correspond to the piecewise-linear function ("cover-function")

$$\varphi_0(\tau) = \begin{cases} (\tau_1 - \tau)/\delta\tau\,, & \tau \in [\tau_0, \tau_1]\,, \\ 0\,, & \tau \notin [\tau_0, \tau_1]\,, \end{cases}$$

$$\varphi_1(\tau) = \begin{cases} (\tau - \tau_{i-1})/\delta\tau\,, & \tau \in [\tau_{i-1}, \tau_i]\,, \\ (\tau_{i+1} - \tau)/\delta\tau\,, & \tau \in [\tau_i, \tau_{i+1}]\,, \quad i = 1, 2, ..., n-1\,, \\ 0\,, & \tau \notin [\tau_{i-1}, \tau_{i+1}]\,, \end{cases} \tag{21}$$

$$\varphi_n(\tau) = \begin{cases} (\tau - \tau_{n-1})/\delta\tau\,, & \tau \in [\tau_{n-1}, \tau_n]\,, \\ 0\,, & \tau \notin [\tau_{n-1}, \tau_n]\,. \end{cases}$$

Assuming

$$p^*(\tau) \cong \tilde{p}^*(\tau) = \sum_{i=0}^{n} p_i^* \varphi_i(\tau)\,, \quad p_i^* \equiv p^*(\tau_i)\,, \tag{22}$$

we will represent the solution (11) as follows:

$$T(\xi, \eta) = \frac{\Lambda}{\delta\tau\sqrt{\pi}} \sum_{i=0}^{n} p_i^* G_i^{(1)}(\xi, \eta) H(\xi)\,, \quad -\infty < \xi < \infty\,, \quad 0 \le \eta < \infty\,, \tag{23}$$

$$G_0^{(1)}(\xi, \eta) = \tau_1 \tilde{G}_1^{(0)}(\xi, \eta) - \tilde{G}_1^{(1)}(\xi, \eta)\,,$$

$$G_i^{(1)}(\xi, \eta) = \tilde{G}_i^{(1)}(\xi, \eta) - \tau_{i-1}\tilde{G}_i^{(0)}(\xi, \eta) + \tau_{i+1}\tilde{G}_{i+1}^{(0)}(\xi, \eta) - \tilde{G}_{i+1}^{(1)}(\xi, \eta)\,, \quad i = 1, 2, ..., n-1\,, \tag{24}$$

$$G_n^{(1)}(\xi, \eta) = \tilde{G}_n^{(1)}(\xi, \eta) - \tau_{n-1}\tilde{G}_n^{(0)}(\xi, \eta)\,,$$

$$\tilde{G}_i^{(1)}(\xi, \eta) = \int_{\tau_{i-1}}^{\tau_i} \tau G(\xi - \tau, \eta)\, d\tau\,, \quad \xi \ge \tau_i\,, \quad i = 1, 2, ..., n\,. \tag{25}$$

Integration in relation (25) leads to the formulas

$$\tilde{G}_i^{(1)}(\xi, \eta) = \frac{2}{3}(\xi - \tau_i)^{3/2} \exp\left[-\frac{\eta^2}{4(\xi - \tau_i)}\right] - \frac{2}{3}(\xi - \tau_{i-1})^{3/2} \times$$

$$\times \exp\left[-\frac{\eta^2}{4(\xi - \tau_{i-1})}\right] + \left(\frac{\eta^2}{6} + \xi\right)\tilde{G}_i^{(0)}(\xi, \eta)\,, \quad \xi \ge \tau_i\,, \quad i = 1, 2, ..., n\,, \tag{26}$$

where the functions $\tilde{G}_i^{(0)}(\xi, \eta)$ have the form of (18).

Taking into account relation (19), from formula (26) for η = 0 we obtain

$$\tilde{G}_i^{(1)}(\xi, 0) = \frac{2}{3}(2\xi + \tau_{i-1})\sqrt{\xi - \tau_{i-1}} - \frac{2}{3}(2\xi + \tau_i)\sqrt{\xi - \tau_i}\,,$$

$$\xi \ge \tau_i\,, \quad i = 1, 2, ..., n\,. \tag{27}$$

By substituting expressions (19) and (27) into formulas (24) we find, from equality (23), the temperature distribution on the surface of the half-space:

$$T(\xi) = \frac{\Lambda}{\delta\tau\sqrt{\pi}} \sum_{i=0}^{n} p_i^* G_i^{(1)}(\xi, 0)\, H(\xi)\,, \quad -\infty < \xi < \infty\,; \tag{28}$$

$$G_0^{(1)}(\xi, 0) = 2\left(\tau_1 - \frac{2}{3}\xi\right)\sqrt{\xi} + \frac{4}{3}(\xi - \tau_1)\sqrt{\xi - \tau_1}\,,$$

$$G_i^{(1)}(\xi, 0) = \frac{4}{3}(\xi - \tau_{i-1})\sqrt{\xi - \tau_{i-1}} + 2\left(\tau_{i-1} - \frac{2}{3}\tau_i + \tau_{i+1} - \frac{4}{3}\xi\right)\sqrt{\xi - \tau_i} +$$

$$+ \frac{4}{3}(\xi - \tau_{i+1})\sqrt{\xi - \tau_{i+1}}\,, \quad i = 1, 2, ..., n-1\,,$$

$$G_n^{(1)}(\xi, 0) = \frac{4}{3}(\xi - \tau_{n-1})\sqrt{\xi - \tau_{n-1}} + 2\left(\tau_{n-1} - \frac{1}{3}\tau_n - \frac{2}{3}\xi\right)\sqrt{\xi - \tau_n}\,. \tag{29}$$

Numerical Analysis and Conclusions. In the case of ideally smooth surfaces of contacting bodies, one most frequently uses the elliptical (Hertz) distribution of the contact pressure [15]

$$p(x) = p_0\sqrt{1 - \frac{(x-a)^2}{a^2}}\,, \quad p_0 = \frac{2P}{\pi a}\,, \quad 0 \le x \le 2a\,. \tag{30}$$

If the working surfaces of the bodies are rough, a superposition of the Hertz (30) and oscillating distributions is taken. We take, as an example, cosine fluctuations of pressure with a wavelength of 5π and an amplitude of $p_0/4$ of the form [8]

$$p(x) = p_0\left[\sqrt{1 - \left(\frac{x-a}{a}\right)^2} - \frac{1}{4}\cos 5\pi\frac{x-a}{a}\right], \quad 0 \le x \le 2a\,. \tag{31}$$

On heavily loaded contacts when the materials of the bodies are in a nearly plastic state the pressure equalizes:

$$p(x) = \frac{\pi}{4}p_0 = \text{const}\,, \quad 0 \le x \le 2a\,. \tag{32}$$

For the constant pressure (32), the temperature field in the half-space is found by direct integration of the solution (11) and (12) using the value (18) of the integral (17). As a result we obtain

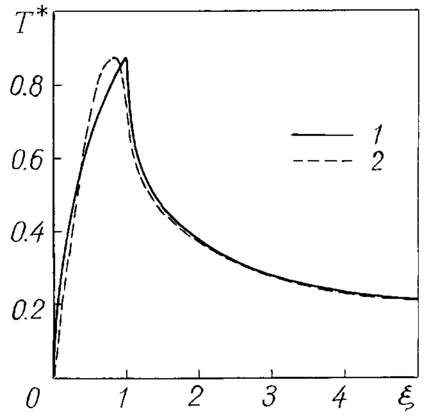

Fig. 1. Distribution of the dimensionless surface temperature T^*: 1) constant distribution of the contact pressure (32); 2) Hertz distribution (30).

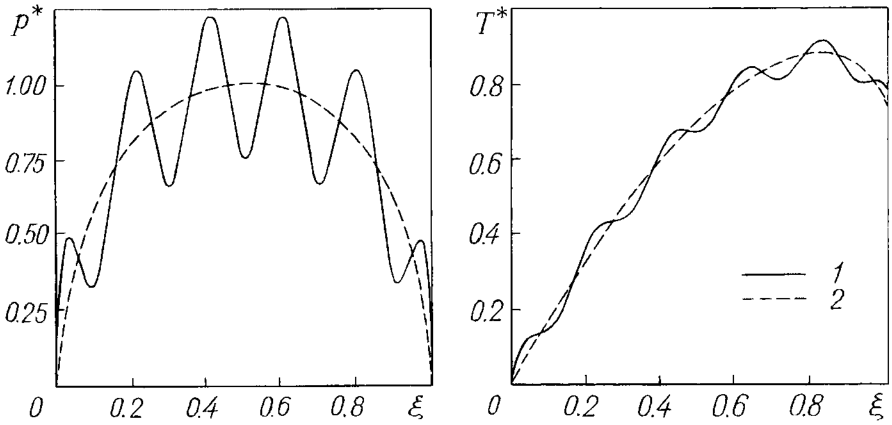

Fig. 2. Distribution of the dimensionless contact pressure p^* and surface temperature T^*: 1) oscillating distribution (31) of the contact pressure; 2) Hertz distribution (30).

$$T(\xi, \eta) = \begin{cases} 0, & -\infty < \xi \le 0, \\ T_{\max}\, \theta(\xi, \eta), & 0 \le \xi \le 1, \\ T_{\max}\, [\theta(\xi, \eta) - \theta(\xi - 1, \eta)], & 1 \le \xi < \infty, \end{cases} \tag{33}$$

where

$$\theta(\xi, \eta) = \sqrt{\xi}\, \exp\left(-\frac{\eta^2}{4\xi}\right) - \eta \frac{\sqrt{\pi}}{2} \operatorname{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right), \quad 0 \le \eta < \infty; \tag{34}$$

$$T_{\max} = \Lambda \sqrt{\pi}/2 \cong 0.886\Lambda, \tag{35}$$

and the constant Λ has the form of (9).

The distribution of the temperature (33)–(35) was used to test the approximate procedure proposed. For the Hertz (30) and oscillating (31) distributions of the contact pressure, the temperature field in the half-space was investigated numerically by formulas (16)–(20) in the case of the step approximation of the pressure (14) and (15) or by formulas (23)–(29) for its piecewise-linear approximation (22).

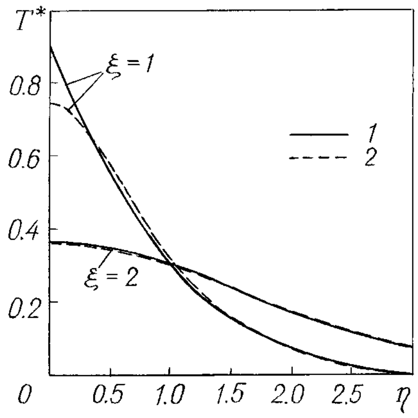

Fig. 3. Change in the dimensionless temperature T^* over the depth η from the surface of a half-space: 1) constant distribution of the contact pressure (32); 2) Hertz distribution (30).

The distribution of the dimensionless surface temperature $T^*(\xi) = T(\xi, 0)/\Lambda$ is shown in Fig. 1. We note that T_{max} of (35) is the maximum temperature at the constant pressure (32), which, as is obvious from the figure, is attained at the point $\xi = 1$ ($x = 2a$ is the point of departure from the heating zone). In the case of the Hertz pressure (30), the maximum value of the surface temperature is attained within the heating region at the point $x = 1.652a$ and is equal to $T_{max} \cong 0.872\Lambda$, which amounts to 98.4% of T_{max} of (35).

Distributions of the dimensionless oscillating (31) and Hertz (30) pressures and the corresponding surface temperatures are shown in Fig. 2. Here it is of interest to note that a 25% difference in the maximum values of these pressures involves just a 6% difference in the corresponding maximum temperatures.

Figure 3 illustrates the change in the dimensionless temperature $T^*(\xi, \eta) = T(\xi, \eta)\Lambda$ depthwise along the normal to the surface of the half-space. It is seen that the temperature under the heating region $0 \le \xi \le 1$ decreases over depth more rapidly that the temperature under the free surface $\xi > 1$. In the cross section $\xi = 1$, the temperature field becomes insignificant for $y > 3d$, where the parameter d is the effective depth of penetration of heat in the corresponding nonstationary problem of heat conduction [8].

NOTATION

T, temperature; $T^*(\xi, \eta) = T(\xi, \eta)/\Lambda$, dimensionless temperature; Λ, constant determined from formula (9) and having the dimensions of temperature; a, halfwidth of the heating region; K, thermal-conductivity coefficient; k, thermal-diffusivity coefficient; V, velocity of motion of the linear heat flux; $q(x)$, intensity of the heat flux; (x, y), axes of the orthogonal Euler coordinate system; f, coefficient of friction; p, contact pressure; p_0, characteristic value of the contact pressure; d, effective depth of heating; Pe, Péclet parameter; P, linear pressing force; $H(\cdot)$, Heaviside unit function; erf(·), probability integral; erfc(·) = 1 − erf(·).

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